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b-AM-compact Operators on Banach Lattices

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Abstract: Several characterizations of b-AM-compact operators are considered in this paper, we show that: 1) If F is an infinite-dimensional Banach lattice, then E is a KB-space if and only if every AM-compact operator from E into F is b-AM-compact. 2) The Banach lattice E is a discrete KB-space if and only if every continuous operator from E into Banach lattice F is b-AM-compact. 3) If the topological dual E' is discrete, then every b-weakly compact operator from Banach E into Banach space X is b-AM-compact. Moreover, following properties about the problems of domination in the class of positive b-AM-compact operators are established: 1) If E and F are two Banach lattices, then for all operators S, $T : E \to F$ such that $0 \leq S \leq T$ and T is b-AM-compact, the operator S is b-AM-compact if and only if the norm of F is order continuous or E' is discrete. 2) If S, T are two operators from E into F with $0 \leq S \leq T$, if T is b-AM-compact, then S^2 is likewise b-AM-compact.

Keywords: Banach lattices; b-AM-compact operator

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1 Introduction

The Riesz spaces considered in this note are assumed to have separating order duals. For unexplained terminology on Riesz space, we refer to [1,2].

Definition 1 A subset A of a Riesz space E is called b-order bounded in E if it is order bounded in $(E^{\sim})^{\sim}$. A Riesz space is said to have property (b) if $A \subset E$ is order bounded whenever A is order bounded in $(E^{\sim})^{\sim}$.

A Riesz space E have property (b) if and only if for each net $\{x_\alpha\}$ in E with $0 \leq x_\alpha \uparrow \leq \hat{x}$ for some $\hat{x} \in (E^{\sim})^{\sim}$, $\{x_\alpha\}$ is order bounded in E.

Note that every perfect Riesz space as well as every order dual has property (b). And every reflexive Banach lattice has property (b). Moreover, every KB-space has property (b) and if $(E^{\sim})^{\sim}$ is retractable on E then E has property (b). On the other hand, considering $A = \{e_n\}$ in c_0, we see that c_0 does not have property (b).

2 Characterizations of the b-AM-compact operators

Let E, F be Banach lattices and X a Banach space.

Definition 2 An operator $T : E \to X$ is called b-AM-compact if T maps each b-order bounded subset of E into a relatively compact subset of X.

The collection of all b-AM-compact operators from E into X is denoted by $C_{\text{b-AM}}(E, X)$. Then $C_{\text{b-AM}}(E, X)$ is a closed subspace of $L(E, X)$, the space of continuous linear operators from E into X. Recall that operators mapping order intervals into relatively compact sets are

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called AM-compact operators and are denoted by $C_{\mathrm{AM}}(E,X)$. Let $C(E,X)$ be the space of compact operators. Clearly, we have $C(E,X) \subseteq C_{\text{b-AM}}(E,X) \subseteq C_{\mathrm{AM}}(E,X)$.

We now give examples to show that these inclusions are proper.

Example 1 (a) Consider the identity operator $I : l_p \to l_p (1 < p < \infty)$. Since l_p is a reflexive Banach lattice, b-order bounded subset of l_p is order bounded and is relatively compact. That is to say, I is b-AM-compact. But I is not compact.

(b) Consider the identity operator $I : c_0 \to c_0$. Since c_0 is a discrete Banach lattice with order continuous norm, I is AM-compact. But it is not b-AM-compact. However as l_1 is a KB-space, $I' : l_1 \to l_1$ is a b-AM-compact operators.

(c) Although, $I : l_1 \to l_1$ is a b-AM-compact operator, $I' : l_\infty \to l_\infty$ can not be b-AM-compact.

Theorem 1 If F is a infinite-dimensional Banach lattice, then E is a KB-space if and only if every AM-compact operator from E into F is b-AM-compact.

Proof If E is a KB-space, E has the property (b), thus every AM-compact operator from E into F is b-AM-compact.

Conversely, if every AM-compact operator from E into F is b-AM-compact, but E is not a KB-space, then it contains a sublattice H which is isomorphic to c_0 (see [2, Theorem 2.4.12]). Let ψ be this isomorphism, it admits a positive extension $\tilde{\psi} : E \to c_0$ (see [3, Theorem 1]).

Since every positive operator $S : c_0 \to F$ is AM-compact, but need not to be b-AM-compact. We consider the operator product $S \circ \tilde{\psi} : E \to c_0 \to F$. It is positive and AM-compact but not b-AM-compact.

Theorem 2 For Banach lattice E the following assertions are equivalent:

1) E is a discrete KB-space;

2) For any Banach lattice F, every continuous operator from E into F is b-AM-compact.

Proof Let E be a discrete and KB-space, T be a continuous operator from E into F, and A be a b-order bounded subset of E. Since every KB-space has property (b), A is order bounded in E, so there exists a positive element $x \in E_+$ with $A \subset [-x, x]$. Since E is discrete with order continuous norm, the order interval $[-x, x]$ is a compact subset of E and it is norm closed. Since the closure of A is a subset of $[-x, x]$, it follows that A is a relatively norm compact subset in E. From T is a continuous operator, $T(A)$ is relatively norm compact subset of F (see [1], Theorem 17.1). So $L(E,F) \subset C_{\text{b-AM}}(E,F)$. On the other hand, $C_{\text{b-AM}}(E,F) \subset L(E,F)$ is satisfied (see [4, Theorem 1.3]), it follows that $L(E,F) = C_{\text{b-AM}}(E,F)$.

Now we assume that $L(E,F) = C_{\text{b-AM}}(E,F)$ holds for every Banach lattice F. Then the identity operator $I : E \to E$ is a b-AM-compact operator, thus it is b-weakly compact, it follows that E is a KB-space (see [5, Proposition 2.10]). From $I : E \to E$ is b-AM-compact, it follows that the order interval in E is relatively compact. Since the order interval in Banach lattice is norm closed, it follows that the order interval in E is compact. Therefore, E is discrete (see [6, Corollary 21.13]).

Recall that an operator $T : E \to X$, mapping each b-order bounded subset of E into a relatively weakly compact subset of X is called a b-weakly compact operator. A nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the subspace generated by x. The vector lattice E is discrete, if it admits a complete disjoint system of

discrete elements.

Theorem 3 If the topological dual E' is discrete, then every b-weakly compact operator from Banach lattice E into Banach space X is b-AM-compact.

Proof Let A be a b-order bounded subset in E. We choose $\hat{x} \in E''_+$ with $A \subseteq [-\hat{x}, \hat{x}]$. $I_{\hat{x}}$ be the principal ideal generated by $\hat{x}$ in E'' and $Y = I_{\hat{x}} \cap E$. Denote by T, the restriction of the operator T to Y. Since T is b-weakly compact, it follows that $T : Y \to X$ is a weakly compact operator. Then $T' : X' \to Y'$ is a weakly compact operator. Since $(Y, \|\cdot\|_\infty)$ is an AM-space, its topological dual Y' is an AL-space. If we denote by B the solid hull of $T'(B_{X'})$ where $B_{X'}$ is the unit ball of X', it follows that each disjoint sequence of B is convergent for the norm of Y' (see [6, Theorem 21.10]).

Now, the inclusion mapping $I : Y \to E$ is a lattice homomorphism, it follows that $I' : E' \to Y'$ is interval-preserving, where Y' is the topological dual of $(Y, \|\cdot\|_\infty)$, hence E' is an ideal of Y'. And each discrete element of E' is a discrete element of Y'. Since the Banach lattice E' is discrete, it follows that $B \subset E'$, then B is contained in the band generated by the discrete element of Y'. Hence, it results that the solid bounded subset B is relatively compact for the norm of Y' (see [6, Theorem 21.15]). So $T'(B_{x'})$ is also relatively compact for the norm of Y'. This shows that the operator $T' : X' \to Y'$ is compact and then T is a compact operator from Y into X. This, together with the fact that A is norm bounded imply that $T(A)$ is relatively norm compact.

The following example shows that the above theorem is not sufficient.

Example 2 There exist a operator $T : l_\infty \to X$ such that T is compact (see [3, Theorem 1]). Hence T is b-weakly compact and b-AM-compact. But the topological dual $(l_\infty)'$ is not discrete.

3 The domination of the b-AM-compact operators

An application of Theorem 3 generalizes this theorem as follows.

Corollary 1 Let E and F be Banach lattices, such that E' is discrete, and let $S, T : E \to F$ be operators with $0 \le S \le T$. If T is b-AM-compact, then so is S.

Theorem 4 Let E and F be Banach lattices, such that for each $\hat{x} \in E''_+$, the vector lattice $(Y_{\hat{x}})'$ is discrete, and let $S, T : E \to F$ be operators with $0 \le S \le T$. If T is b-AM-compact, then so is S.

Proof Let S and T be operators from E into F such that $0 \le S \le T$ and T is b-AM-compact. It is clear that S is b-AM-compact if and only if for each $\hat{x} \in E''_+$, the restriction $S\mid_{Y_{\hat{x}}}$ from $Y_{\hat{x}}$ into F is compact, where $Y_{\hat{x}} = I_{\hat{x}} \cap E$ and $I_{\hat{x}}$ is the principal ideal generated by $\hat{x}$ in E''. Since $T\mid_{Y_{\hat{x}}}$ from $Y_{\hat{x}}$ into F is compact, $0 \le S\mid_{Y_{\hat{x}}} \le T\mid_{Y_{\hat{x}}}$ and $(Y_{\hat{x}})'$ is discrete with an order continuous norm, it follows that $S\mid_{Y_{\hat{x}}}$ is compact (see [3, Theorem 1]).

Theorem 5 Let E and F be Banach lattices. Then the following statements are equivalent.

1) For all operators S, $T : E \to F$ such that $0 \le S \le T$ and T is b-AM-compact, the operator S is b-AM-compact;

2) One of the following conditions holds:

a) The norm of F is order continuous; b) E' is discrete.

Proof 2)-a)$\Rightarrow$ 1) Let A of E be a b-order bounded subset with $A \subseteq [-\hat{x}, \hat{x}]$, $I_{\hat{x}}$ be the principal ideal generated by $\hat{x}$ in E''_+ and $Y = I_{\hat{x}} \cap E$. Denote by T, the restriction of the operator T to F. Since $(Y, \|\cdot\|_\infty)$ is an AM-space, its topological dual Y' is an AL-space, together with the fact that F has order continuous norm, it follows that $S: Y \to F$ is a compact operator (see [1, Theorem 16.20]), and $S: E \to F$ is b-AM-compact.

2)-b)$\Rightarrow$ 1) It is just the Corollary 1.

1)$\Rightarrow$ 2) Assume that either of the conditions a) and b) is true, Theorem 2.10 of [7] implies the existence of two operators S and T from E into F such that $0 \le S \le T$ and T is compact, the operator S is not AM-compact.

Theorem 6 Consider the scheme of operator $E \overset{S_1}{\to} G \overset{S_2}{\to} X$, where E and G are Banach lattices, if S_1 is dominated by a b-AM-compact operators, and S_2 is dominated by a o-weakly compact operator, then S_2S_1 is b-AM-compact.

Proof Since S_2 is dominated by a o-weakly operator, it follows that S_2 is a o-weakly operator (see [1, Corollary 18.2]). Thus S_2 admits a factorization through Banach lattice F with order continuous norm (see [2, Theorem 3.4.6]). Clearly, $QS_1 : E \to F$ is dominated by a b-AM-compact operator. Since the norm on F is order continuous, it follows from Theorem 5 that $QS_1 : E \to F$ is a b-AM-compact operator. Consequently, $S_2S_1 = S(QS_1)$ is an b-AM-compact operator.

Corollary 2 Let S, T be operators from E into F with $0 \le S \le T$. If T is b-AM-compact then S^2 is likewise b-AM-compact.

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Banach格上的b-AM-紧算子

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摘 要: 本文对Banach格上的b-AM-紧算子进行了描述，得到了如下三个结论：1) 如果Banach格F是无限维的，则E是KB-空间当且仅当每个从E到F的AM-紧算子是b-AM-紧算子。2) Banach格E是离散的KB-空间当且仅当每个从E到F的连续算子是b-AM-紧算子。3) 如果E'是离散的，则每个从E到F的b-弱紧算子是b-AM-紧算子。其次给出了b-AM-紧算子的控制性质，得到如下两个结论：1) 如果E和F是两个Banach格，算子$S, T : E \to F$满足$0 \le S \le T$且T是b-AM-紧算子，则算子S是b-AM-紧算子当且仅当F具有序连续范数或者E'是离散空间。2) 如果S, T是从E到F的算子满足$0 \le S \le T$，如果T是b-AM-紧算子，则S^2也是b-AM-紧算子。

关键词: Banach格；b-AM-紧算子